week 1

RADIATIVE TRANSFER

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DOVER PUBLICATIONS, INC.
NEW YORK

I

THE EQUATION OF TRANSFER

1. Introduction

IN this chapter we shall define the fundamental quantities which the subject of Radiative Transfer deals with and derive the basic equation—the equation of transfer—which governs the radiation field in a medium which absorbs, emits, and scatters radiation. In formulating the various concepts and equations we shall not aim at the maximum generality possible but limit ourselves, rather, by the situations which the problems considered in this book actually require.

The chapter also includes a classification and discussion of the various types of problems which will be treated in this book.

2. Definitions

2.1. *The specific intensity*

The analysis of a radiation field often requires us to consider the amount of radiant energy, dE_ν, in a specified frequency interval $(\nu, \nu+d\nu)$ which is transported across an element of area $d\sigma$ and in directions confined to an element of solid angle $d\omega$, during a time dt (see Fig. 1). This energy, dE_ν, is expressed in terms of the *specific intensity* (or, more simply, *the intensity*), I_ν, by

$$dE_\nu = I_\nu \cos\vartheta \, d\nu d\sigma d\omega dt, \tag{1}$$

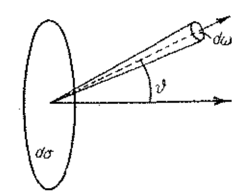

FIG. 1.

where ϑ is the angle which the direction considered makes with the outward normal to $d\sigma$. The construction we have used here defines also a *pencil of radiation*.

It follows from the definition of intensity that in a medium which absorbs, emits, and scatters radiation, I_ν may be expected to vary from point to point and also with direction through every point. Thus, for a general radiation field, we may write

$$I_\nu \equiv I_\nu(x, y, z; l, m, n; t), \tag{2}$$

where (x, y, z) and the direction cosines (l, m, n) define the point and the direction to which I_ν refers.

A radiation field is said to be *isotropic* at a point, if the intensity is independent of direction at that point. And if the intensity is the same at all points and in all directions, the radiation field is said to be *homogeneous* and isotropic.

The case of greatest interest in astrophysical (and terrestrial) contexts is that of an atmosphere stratified in parallel planes in which all the physical properties are invariant over a plane. In this case we can write

$$I_\nu \equiv I_\nu(z, \vartheta, \varphi; t), \tag{3}$$

where z denotes the height measured normal to the plane of stratification and ϑ and φ are the polar and azimuthal angles, respectively. If I_ν should be further independent of φ we have a field which has *axial symmetry* about the z-axis.

Another case of interest which also arises in practice is that of *spherical symmetry* when

$$I_\nu \equiv I_\nu(r, \vartheta; t), \tag{4}$$

where r is the distance from the centre of symmetry and ϑ is the inclination of the direction considered to the radius vector.

The intensity I_ν integrated over all the frequencies is denoted by I and is called the *integrated intensity*; thus

$$I = \int_0^\infty I_\nu \, d\nu. \tag{5}$$

While for most purposes the intensity $I_\nu(x, y, z; l, m, n)$ sufficiently characterizes a radiation field, it is important to note that further parameters describing the state of polarization of the radiation field must be specified before we can regard the description of the field as really complete. We shall consider the characterization of these further parameters in § 15.

2.2. *The net flux*

Equation (1) gives the energy in the frequency interval $(\nu, \nu+d\nu)$ which flows across an element area of $d\sigma$ in a direction which is inclined at an angle ϑ to its outward normal and confined to an element of solid angle $d\omega$. The net flow in all directions is therefore given by

$$d\nu d\sigma dt \int I_\nu \cos\vartheta \, d\omega, \tag{6}$$

where the integration is to be effected over all solid angles. The quantity

$$\pi F_\nu = \int I_\nu \cos\vartheta \, d\omega \tag{7}$$

which occurs in the expression (6) is called the *net flux* and defines the rate of flow of radiant energy across $d\sigma$ per unit area and per unit frequency interval.

For a system of polar coordinates with the z-axis in the direction of the outward normal to $d\sigma$

$$d\omega = \sin\vartheta \, d\vartheta d\varphi, \tag{8}$$

and the expression for the net flux can be written in the form

$$\pi F_\nu = \int_0^\pi \int_0^{2\pi} I_\nu(\vartheta, \varphi) \sin\vartheta \cos\vartheta \, d\vartheta d\varphi. \tag{9}$$

As F_ν has been defined, it depends on the direction of the outward normal to the elementary surface across which the flow of radiant energy has been considered. However, this dependence of the flux on direction is simple and is of the nature of a vector. For, considering the flux across a surface the direction cosines of whose normal are l, m, and n, we have

$$\pi F_{\nu; l,m,n} = \int I_\nu(l', m', n') \cos\Theta \, d\omega, \tag{10}$$

where Θ is the angle between the directions (l', m', n') and (l, m, n). Hence

$$\pi F_{\nu; l,m,n} = \int I_\nu(l', m', n')(ll' + mm' + nn') \, d\omega, \tag{11}$$

or

$$F_{\nu; l,m,n} = lF_{\nu;x} + mF_{\nu;y} + nF_{\nu;z}, \tag{12}$$

where $F_{\nu;x}$, $F_{\nu;y}$, and $F_{\nu;z}$ define the fluxes across surfaces normal to the x, y, and z directions, respectively.

For a radiation field which has an axial symmetry the expression for F_ν along the axis of symmetry is

$$F_\nu = 2 \int_0^\pi I_\nu(\vartheta) \sin\vartheta \cos\vartheta \, d\vartheta. \tag{13}$$

2.3. *The density of radiation*

The energy density $u_\nu \, d\nu$ of the radiation in the frequency interval $(\nu, \nu+d\nu)$ at any given point is the amount of radiant energy per unit volume, in the stated frequency interval, which is in course of transit in the immediate neighbourhood of the point considered.

To find the expression for the energy density at a point P we construct around P an infinitesimal volume v with a convex bounding surface σ. We next surround v by another convex surface Σ such that the linear dimensions of Σ are large compared with those of σ; nevertheless, we arrange that the volume element enclosed by Σ is still so small that we can regard the intensity in any given direction as the same for all points inside Σ.

Now all the radiation traversing the volume v must have crossed some element of the surface Σ. Let $d\Sigma$ be such an element; further let

Θ and ϑ denote the angles which the normals to $d\Sigma$ and an element $d\sigma$ of σ make with the line joining the two elements. The energy flowing across $d\Sigma$ which also flows across $d\sigma$ is

$$I_\nu \cos\Theta\, d\Sigma d\omega' d\nu = I_\nu\, d\nu \frac{\cos\vartheta \cos\Theta\, d\sigma d\Sigma}{r^2} \tag{14}$$

since the solid angle $d\omega'$ subtended by $d\sigma$ at $d\Sigma$ is

$$d\sigma \cos\vartheta / r^2,$$

where r is the distance between $d\sigma$ and $d\Sigma$. If l is the length traversed by the pencil of radiation considered through the volume element v, the radiation (14) incident on $d\sigma$ per unit time will have traversed the element in a time l/c, where c denotes the velocity of light. The contribution to the total amount of radiant energy in course of transit through v by the pencil of radiation considered is

$$I_\nu\, d\nu \frac{\cos\vartheta \cos\Theta\, d\sigma d\Sigma}{r^2} \frac{l}{c} = \frac{1}{c} I_\nu\, dv d\nu d\omega, \tag{15}$$

where
$$d\omega = d\Sigma \cos\Theta / r^2$$

is the solid angle subtended by $d\Sigma$ at P and

$$dv = l\, d\sigma \cos\vartheta$$

is the volume intercepted in v by the pencil of radiation. Therefore the total energy in the frequency interval $(\nu, \nu+d\nu)$ in course of transit through v, due to the radiation coming from all directions, is obtained by integrating (15) over all v and ω: thus,

$$\frac{d\nu}{c} \int dv \int d\omega\, I_\nu = \frac{v}{c}\, d\nu \int I_\nu\, d\omega. \tag{16}$$

Hence
$$u_\nu = \frac{1}{c} \int I_\nu\, d\omega. \tag{17}$$

The *integrated energy density*, u, is similarly given in terms of the integrated intensity I; thus,

$$u = \int_0^\infty u_\nu\, d\nu = \frac{1}{c} \int I\, d\omega. \tag{18}$$

It is often convenient to introduce the *average intensity*

$$J_\nu = \frac{1}{4\pi} \int I_\nu\, d\omega, \tag{19}$$

which is related to the energy density by

$$u_\nu = \frac{4\pi}{c} J_\nu. \tag{20}$$

For an axially symmetric radiation field (cf. eq. [13])

$$J_\nu = \tfrac{1}{2} \int_0^\pi I_\nu \sin\vartheta\, d\vartheta. \tag{21}$$

3. Absorption coefficient. True absorption and scattering. Phase function

A pencil of radiation traversing a medium will be weakened by its interaction with matter. If the specific intensity I_ν therefore becomes $I_\nu + dI_\nu$ after traversing a thickness ds in the direction of its propagation, we write

$$dI_\nu = -\kappa_\nu \rho I_\nu\, ds, \tag{22}$$

where ρ is the density of the material. The quantity κ_ν introduced in this manner defines the *mass absorption coefficient* for radiation of frequency ν. Now it should not be assumed that this reduction in intensity, which a pencil of radiation in passing through matter experiences, is necessarily lost to the radiation field. For it can very well happen that the energy lost from the incident pencil may all reappear in other directions as *scattered radiation*. In general we may, however, expect that only a part of the energy lost from an incident pencil will reappear as scattered radiation in other directions and that the remaining part will have been 'truly' absorbed in the sense that it represents the transformation of radiation into other forms of energy (or even of radiation of other frequencies). We shall therefore have to distinguish between *true absorption* and *scattering*.

Considering first the case of scattering, we say that a material is characterized by a *mass scattering coefficient* κ_ν if from a pencil of radiation incident on an element of mass of cross-section $d\sigma$ and height ds, energy is scattered from it at the rate

$$\kappa_\nu \rho\, ds \times I_\nu \cos\vartheta\, d\nu d\sigma d\omega \tag{23}$$

in all directions. Since the mass of the element is

$$dm = \rho \cos\vartheta\, d\sigma ds, \tag{24}$$

we can also write
$$\kappa_\nu I_\nu\, dm d\nu d\omega. \tag{25}$$

It is now evident that to formulate quantitatively the concept of scattering we must specify in addition the angular distribution of the scattered radiation (25). We shall therefore introduce a *phase function* $p(\cos\Theta)$ such that

$$\kappa_\nu I_\nu p(\cos\Theta) \frac{d\omega'}{4\pi}\, dm d\nu d\omega \tag{26}$$

gives the rate at which energy is being scattered into an element of solid angle $d\omega'$ and in a direction inclined at an angle Θ to the direction of incidence of a pencil of radiation on an element of mass dm. According to (26) the rate of loss of energy from the incident pencil due to scattering in all directions is

$$\kappa_\nu I_\nu \, dm d\nu d\omega \int p(\cos\Theta)\frac{d\omega'}{4\pi}; \tag{27}$$

this agrees with (25) if

$$\int p(\cos\Theta)\frac{d\omega'}{4\pi} = 1, \tag{28}$$

i.e. if the phase function is *normalized to unity*.

Returning to the general case when both scattering and true absorption are present, we shall still write for the scattered energy the *same expression* (26). But in this case (in contrast to the case of scattering only) the total loss of energy from the incident pencil must be less than (25); accordingly

$$\int p(\cos\Theta)\frac{d\omega'}{4\pi} = \varpi_0 \leqslant 1. \tag{29}$$

Thus the general case differs from the case of pure scattering only by the fact that the phase function is not normalized to unity.

It is evident from our definitions that ϖ_0 represents the fraction of the light lost from an incident pencil due to scattering, while $(1-\varpi_0)$ represents the remaining fraction which has been transformed into other forms of energy (or of radiation of other wave-lengths).

We shall refer to ϖ_0 as the *albedo for single scattering*. Moreover, when $\varpi_0 = 1$ we shall say that we have a *conservative case* of *perfect scattering*: perfect scattering is, in our present context, the analogue of the concept of conservatism in dynamics.

The simplest example of a phase function is

$$p(\cos\Theta) = \text{constant} = \varpi_0. \tag{30}$$

In this case the radiation scattered by each element of mass is *isotropic*. Next to this isotropic case greatest interest is attached to *Rayleigh's phase function* (cf. § 16)

$$p(\cos\Theta) = \tfrac{3}{4}(1+\cos^2\Theta). \tag{31}$$

This phase function is normalized to unity so that this is an example of a conservative case of perfect scattering. Another phase function which is of particular interest in problems relating to planetary illumination is

$$p(\cos\Theta) = \varpi_0(1+x\cos\Theta) \quad (-1 \leqslant x \leqslant +1). \tag{32}$$

In general we may suppose that the phase function can be expanded as a series in Legendre polynomials of the form

$$p(\cos\Theta) = \sum_{l=0}^{\infty} \varpi_l P_l(\cos\Theta), \tag{33}$$

where the ϖ_l's are constants. In practice the series on the right-hand side is a terminating one with only a finite number of terms.

4. The emission coefficient

The *emission coefficient* j_ν is defined in such a way that an element of mass dm emits in directions confined to an element of solid angle $d\omega$, in the frequency interval $(\nu, \nu+d\nu)$ and in time dt, an amount of radiant energy given by

$$j_\nu \, dm d\omega d\nu dt. \tag{34}$$

In the case of a medium which scatters radiation (not necessarily with an albedo $\varpi_0 = 1$) there will be a contribution to the emission coefficient from the scattering of radiation from all other directions into the pencil of directions considered. Thus it follows from (26) that the scattering of a pencil of radiation from a direction (ϑ', φ') (say) contributes to a pencil in the direction (ϑ, φ), energy at the rate

$$\kappa_\nu \, dm d\nu d\omega \; p(\vartheta, \varphi; \vartheta', \varphi') I_\nu(\vartheta', \varphi') \frac{\sin\vartheta' \, d\vartheta' d\varphi'}{4\pi}, \tag{35}$$

where we have written $p(\vartheta, \varphi; \vartheta', \varphi')$ to denote the phase function for the angle between the directions specified by (ϑ, φ) and (ϑ', φ'). Hence the contribution, $j_\nu^{(s)}$, to the emission coefficient by scattering alone is

$$j_\nu^{(s)}(\vartheta, \varphi) = \kappa_\nu \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} p(\vartheta, \varphi; \vartheta', \varphi') I_\nu(\vartheta', \varphi') \sin\vartheta' \, d\vartheta' d\varphi'. \tag{36}$$

We may expect that in general there will be contributions to the emission coefficient from causes other than scattering. When this is *not* the case we shall say that we have a *scattering atmosphere*. In other words, for a scattering atmosphere

$$j_\nu \equiv j_\nu^{(s)}. \tag{37}$$

(Note that a scattering atmosphere does *not* imply that we have a case of perfect scattering.)

A case which is in some sense the opposite of a scattering atmosphere is that of an atmosphere in *local thermodynamic equilibrium*. In this latter case, it is assumed that the circumstances are such that we can define at each point in the atmosphere a *local temperature* T such that the emission coefficient at that point is given in terms of the absorption coefficient by Kirchhoff's law: i.e. at each point we have the relation

$$j_\nu = \kappa_\nu B_\nu(T), \tag{38}$$

where

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} \tag{39}$$

is the Planck function (k and h are the Boltzmann and Planck constants, respectively).

5. The source function

The ratio of the emission to the absorption coefficient plays an important role in the subsequent developments of the theory. It is called the *source function*. We shall denote it by $\mathfrak{J}_\nu$. Thus

$$\mathfrak{J}_\nu = \frac{j_\nu}{\kappa_\nu}. \tag{40}$$

According to equations (36) and (37), for a scattering atmosphere

$$\mathfrak{J}_\nu(\vartheta,\varphi) = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} p(\vartheta,\varphi;\vartheta',\varphi')I_\nu(\vartheta',\varphi')\sin\vartheta'\,d\vartheta' d\varphi', \tag{41}$$

while for an atmosphere in local thermodynamic equilibrium

$$\mathfrak{J}_\nu = B_\nu(T). \tag{42}$$

6. The equation of transfer

We shall now derive the fundamental equation which governs the variation of intensity in a medium characterized by an absorption coefficient κ_ν and an emission coefficient j_ν. (It should be noted that the emission coefficient can itself depend on the radiation field as will be the case, for example, in a scattering atmosphere.) For this purpose consider a small cylindrical element of cross-section $d\sigma$ and height ds in the medium. From the definition of intensity, it now follows that the difference in the radiant energy in the frequency interval $(\nu, \nu+d\nu)$ crossing the two faces normally, in a time dt and confined to an element of solid angle, is given by

$$\frac{dI_\nu}{ds}\,dsd\nu d\sigma d\omega dt. \tag{43}$$

This difference in energy must arise from the excess of emission over absorption in the frequency interval and element of solid angle considered. Now the amount absorbed is (cf. eq. [23])

$$\kappa_\nu \rho\, ds \times I_\nu\, d\nu d\sigma d\omega dt, \tag{44}$$

while the amount emitted is

$$j_\nu \rho\, d\sigma ds d\nu d\omega dt. \tag{45}$$

Counting up the gains and losses in the pencil of radiation during its traversal of the cylinder, we have

$$\frac{dI_\nu}{ds} = -\kappa_\nu \rho I_\nu + j_\nu \rho. \tag{46}$$

In terms of the source function $\mathfrak{J}_\nu$ (eq. [40]) we can rewrite this equation in the form

$$-\frac{dI_\nu}{\kappa_\nu \rho\, ds} = I_\nu - \mathfrak{J}_\nu. \tag{47}$$

This is *the equation of transfer*.

For a scattering atmosphere and an atmosphere in local thermodynamic equilibrium the source functions are given by equations (41) and (42).

In a Cartesian system of coordinates the equation of transfer can be written in the form

$$-\frac{1}{\kappa_\nu \rho}\left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}+n\frac{\partial}{\partial z}\right)I_\nu(x,y,z;l,m,n) = I_\nu(x,y,z;l,m,n)-\mathfrak{J}_\nu(x,y,z;l,m,n). \tag{48}$$

Since the source function is functionally dependent on the intensity at a point, the equation of transfer is generally an *integro-differential equation*. We shall presently have examples of such integro-differential equations.

7. The formal solution of the equation of transfer

In our further discussion in this and the following chapters it is convenient to suppress the suffixes ν to the various quantities: no ambiguity is likely to arise from this. Thus we shall write the equation of transfer (47) in the form

$$-\frac{dI}{\kappa\rho\, ds} = I - \mathfrak{J}. \tag{49}$$

The formal solution of equation (49) is readily written down. We have (see Fig. 2)

$$I(s) = I(0)e^{-\tau(s,0)} + \int_0^s \mathfrak{J}(s')e^{-\tau(s,s')}\kappa\rho\, ds', \tag{50}$$

where $\tau(s,s')$ is the *optical thickness* of the material between the points s and s'; thus

$$\tau(s,s') = \int_{s'}^{s} \kappa\rho\, ds. \tag{51}$$

The physical meaning of the solution (50) is clear: It expresses the fact that the intensity at any point and in a given direction results from the emission at all anterior points, s', reduced by the factor $e^{-\tau(s,s')}$ to allow for the absorption by the intervening matter.

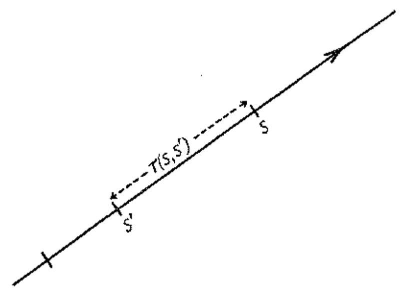

Fig. 2.

Sometimes it is convenient not to stop the range of integration over the source function at some definite point as we have done in equation (50) but write instead

$$I(s) = \int_{-\infty}^{s} \mathfrak{J}(s')e^{-\tau(s,s')}\kappa\rho\, ds'. \tag{52}$$

If the medium extends to $-\infty$ in the direction $\mathbf{s}$, no ambiguity arises by expressing the intensity $I(s)$ in the manner of equation (52). But if for decreasing s' we should encounter a 'radiating surface', for example, then we should stop the integration over s' at this point and add an extra term (as in eq. [50]) to take into account the intensity radiated by the surface. We shall take equation (52) to mean this.

It is of course clear that equations (50) and (52) do not in any real sense 'solve' the equation of transfer. But it is clear that if the source function should depend on the intensity in some specified way, then we can convert the formal solution (52) into an *integral equation* for the source function. We shall encounter examples of such integral equations in § 11.

8. The equation of transfer for a scattering atmosphere. The flux integral for conservative cases

For a scattering atmosphere the source function can be written in the form (cf. eq. [41])

$$\mathfrak{J}(\mathbf{r},\mathbf{s}) = \frac{1}{4\pi}\int p(\mathbf{s},\mathbf{s}')I(\mathbf{r},\mathbf{s}')\, d\omega_{\mathbf{s}'}, \tag{53}$$

where $\mathbf{s}$ is a unit vector specifying some direction through a point $\mathbf{r}$.

The equation of transfer (48) can accordingly be written, in this case, in the form

$$-\frac{1}{\kappa\rho}(\mathbf{s}\,.\,\mathrm{grad})I(\mathbf{r},\mathbf{s}) = I(\mathbf{r},\mathbf{s}) - \frac{1}{4\pi}\int p(\mathbf{s},\mathbf{s}')I(\mathbf{r},\mathbf{s}')\, d\omega_{\mathbf{s}'}. \tag{54}$$

Integrating this equation over all directions s we have

$$-\frac{1}{\kappa\rho}\int (\mathbf{s}\,.\,\mathrm{grad})I(\mathbf{r},\mathbf{s})\, d\omega_{\mathbf{s}} = \int I(\mathbf{r},\mathbf{s})\, d\omega_{\mathbf{s}} - \frac{1}{4\pi}\int\int p(\mathbf{s},\mathbf{s}')I(\mathbf{r},\mathbf{s}')\, d\omega_{\mathbf{s}}\, d\omega_{\mathbf{s}'}. \tag{55}$$

It is evident that the quantity on the left-hand side is the divergence of the vector $\pi \boldsymbol{F}$ whose components are the fluxes parallel to the x-, y-, and z-axes. And the first term on the right-hand side is clearly $4\pi J$ (cf. eq. [19]); the second term is also expressible in terms of J by evaluating the integral of $p(\mathbf{s},\mathbf{s}')$ over the directions $\mathbf{s}$ first (cf. eq. [29]). We therefore have

$$-\frac{1}{4\kappa\rho}\,\mathrm{div}\,\boldsymbol{F} = (1-\varpi_0)J, \tag{56}$$

where ϖ_0 is the albedo for single scattering.

In cases of perfect scattering $\varpi_0 = 1$ and

$$\mathrm{div}\,\boldsymbol{F} = 0. \tag{57}$$

This represents the *flux integral* for conservative problems.

For a plane-parallel atmosphere equation (57) reduces to

$$\frac{dF_z}{dz} = 0 \quad \text{or} \quad F_z = \text{constant}; \tag{58}$$

the net flux normal to the plane of stratification is therefore constant through the atmosphere.

For radiation fields having spherical symmetry the flux integral reduces to

$$F_r = \frac{F_0}{r^2}, \tag{59}$$

where F_0 is a constant; similarly for fields having cylindrical symmetry

$$F_r = \frac{F_0}{r}. \tag{60}$$

9. The equation of transfer for plane-parallel problems

In problems of radiative transfer in plane-parallel atmospheres it is convenient to measure linear distances normal to the plane of stratification.

If z is this distance, the equation of transfer becomes

$$-\cos\vartheta\,\frac{dI(z,\vartheta,\varphi)}{\kappa\rho\,dz} = I(z,\vartheta,\varphi)-\mathfrak{J}(z,\vartheta,\varphi), \tag{61}$$

where ϑ denotes the inclination to the outward normal and φ the azimuth referred to a suitably chosen x-axis.

Introducing the *normal optical* thickness

$$\tau = \int_z^\infty \kappa\rho\,dz, \tag{62}$$

measured from the boundary *inward* we have

$$\mu\,\frac{dI(\tau,\mu,\varphi)}{d\tau} = I(\tau,\mu,\varphi)-\mathfrak{J}(\tau,\mu,\varphi). \tag{63}$$

In equation (63) we have further let $\mu = \cos\vartheta$.

Equation (63) is the standard form of the equation of transfer for plane-parallel atmospheres.

In considering transfer problems in plane-parallel atmospheres we shall distinguish two cases: (i) *the semi-infinite atmosphere* which is bounded on one side ($\tau = 0$) and extends to infinity in the direction $\tau\to\infty$; and (ii) *the finite atmosphere* which is bounded on two sides at $\tau = 0$ and at $\tau = \tau_1$ (say).

In the case of an atmosphere with a finite optical thickness the formal solution (50) reduces to

$$I(\tau,+\mu,\varphi) = I(\tau_1,\mu,\varphi)e^{-(\tau_1-\tau)/\mu}+\int_\tau^{\tau_1}\mathfrak{J}(t,\mu,\varphi)e^{-(t-\tau)/\mu}\,\frac{dt}{\mu}\quad (1\geqslant\mu>0), \tag{64}$$

and

$$I(\tau,-\mu,\varphi) = I(0,-\mu,\varphi)e^{-\tau/\mu}+\int_0^{\tau}\mathfrak{J}(t,-\mu,\varphi)e^{-(\tau-t)/\mu}\,\frac{dt}{\mu}\quad (1\geqslant\mu>0), \tag{65}$$

giving respectively the *outward* and the *inward* intensities at each level. In particular for the *emergent intensities* we have

$$I(0,+\mu,\varphi) = I(\tau_1,\mu,\varphi)e^{-\tau_1/\mu}+\int_0^{\tau_1}e^{-t/\mu}\mathfrak{J}(t,+\mu,\varphi)\,\frac{dt}{\mu}, \tag{66}$$

and $$I(\tau_1,-\mu,\varphi) = I(0,-\mu,\varphi)e^{-\tau_1/\mu}+\int_0^{\tau_1}e^{-(\tau_1-t)/\mu}\mathfrak{J}(t,-\mu,\varphi)\,\frac{dt}{\mu}. \tag{67}$$

In the case of a semi-infinite atmosphere the foregoing equations reduce to

$$I(\tau,+\mu,\varphi) = \int_\tau^\infty \mathfrak{J}(t,+\mu,\varphi)e^{-(t-\tau)/\mu}\,\frac{dt}{\mu}, \tag{68}$$

$$I(\tau,-\mu,\varphi) = I(0,-\mu,\varphi)e^{-\tau/\mu}+\int_0^\tau \mathfrak{J}(t,-\mu,\varphi)e^{-(\tau-t)/\mu}\,\frac{dt}{\mu}, \tag{69}$$

and $$I(0,+\mu,\varphi) = \int_0^\infty \mathfrak{J}(t,+\mu,\varphi)e^{-t/\mu}\,\frac{dt}{\mu}. \tag{70}$$

10. Plane-parallel scattering atmospheres. The K-integral

For a plane-parallel scattering atmosphere the equation of transfer can be written in the form (cf. eq. [41])

$$\mu\,\frac{dI(\tau,\mu,\varphi)}{d\tau} = I(\tau,\mu,\varphi)-\frac{1}{4\pi}\int_{-1}^{+1}\int_0^{2\pi} p(\mu,\varphi;\mu',\varphi')I(\tau,\mu',\varphi')\,d\mu'd\varphi'. \tag{71}$$

In conservative cases ($\varpi_0 = 1$) equation (71) admits the flux integral (eq. [58])

$$F = \text{constant}, \tag{72}$$

where πF represents the flux of radiation normal to the plane of stratification.

There is another integral of importance which conservative problems generally admit. This is the so-called K-integral and can be obtained in the following manner: Multiplying equation (71) by μ and integrating over all solid angles we have

$$\frac{d}{d\tau}\int_{-1}^{+1}\int_0^{2\pi} I(\tau,\mu,\varphi)\mu^2\,d\mu d\varphi$$
$$= \pi F-\frac{1}{4\pi}\int_{-1}^{+1}\int_0^{2\pi} d\mu'd\varphi'\,I(\tau,\mu',\varphi')\int_{-1}^{+1}\int_0^{2\pi} d\mu d\varphi\;\mu p(\mu,\varphi;\mu',\varphi'). \tag{73}$$

Now we shall suppose that $p(\cos\Theta)$ can be expanded, as in equation (33), in a series in Legendre polynomials. Then

$$p(\mu,\varphi;\mu',\varphi') = \sum \varpi_l P_l[\mu\mu'+(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos(\varphi-\varphi')], \tag{74}$$

where it may be recalled that $\varpi_0 = 1$. Expanding P_l in equation (74) by the addition theorem for spherical harmonics we readily find that

$$\frac{1}{4\pi}\int_{-1}^{+1}\int_0^{2\pi} p(\mu,\varphi;\mu',\varphi')\mu\,d\mu d\varphi = \tfrac{1}{2}\varpi_1\mu'\int_{-1}^{+1}\mu^2\,d\mu = \tfrac{1}{3}\varpi_1\mu'. \tag{75}$$

Hence equation (73) reduces to

$$\frac{1}{4\pi}\frac{d}{d\tau}\int_{-1}^{+1}\int_{0}^{2\pi} I(\tau,\mu,\varphi)\mu^2\,d\mu d\varphi = \tfrac{1}{4}(1-\tfrac{1}{3}\varpi_1)F. \tag{76}$$

Now writing

$$K(\tau) = \frac{1}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi} I(\tau,\mu,\varphi)\mu^2\,d\mu d\varphi, \tag{77}$$

we have

$$\frac{dK}{d\tau} = \tfrac{1}{4}(1-\tfrac{1}{3}\varpi_1)F, \tag{78}$$

or, since F is a constant,

$$K = \tfrac{1}{4}F[(1-\tfrac{1}{3}\varpi_1)\tau+Q], \tag{79}$$

where Q is a constant. This is the K-integral.

It is of interest also to notice that in conservative cases the equation of transfer admits a solution of the form

$$I(\tau,\mu) = \text{constant}\left(\tau+\frac{\mu}{1-\frac{1}{3}\varpi_1}\right). \tag{80}$$

For, inserting this form for $I(\tau,\mu)$ in equation (71) and remembering that $\varpi_0 = 1$, we readily verify that the equation is in fact satisfied. Normalizing the solution (80) to yield a net flux πF we have

$$I(\tau,\mu) = \tfrac{3}{4}F[(1-\tfrac{1}{3}\varpi_1)\tau+\mu]. \tag{81}$$

It should be emphasized again that equation (81) does not represent the solution of any physical problem we have formulated. But we shall see (Chap. III, § 25) that there are physical problems for which the solutions tend to (81) as $\tau \to \infty$. The solution (81) is also useful in certain other contexts (Chap. IV, § 29.3).

11. Problems in semi-infinite plane-parallel atmospheres with a constant net flux

We have seen that in conservative cases of perfect scattering the equation of transfer admits the flux integral (72). Because of this constancy of net flux, a type of problem which arises in these contexts is that of a semi-infinite plane-parallel atmosphere with no incident radiation and with a constant net flux, πF, of radiation flowing through the atmosphere normal to the plane of stratification. The particular importance of this type of problem for astrophysics arises from the fact that in *stellar atmospheres* (idealized as plane-parallel atmospheres) the constant net flux is provided by the radiation coming from the 'deep interior'.